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Superconvergent biquadratic finite volume element method for two-dimensional Poisson's equations

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ABSTRACT

In this paper, a kind of biquadratic finite volume element method is presented for two-dimensional Poisson's equations by restricting the optimal stress points of biquadratic interpolation as the vertices of control volumes. The method can be effectively implemented by alternating direction technique. It is proved that the method has optimal energy norm error estimates. The superconvergence of numerical gradients at optimal stress points is discussed and it is proved that the method has also superconvergence displacement at nodal points by a modified dual argument technique. Finally, a numerical example verifies the theoretical results and illustrates the effectiveness of the method.

1. Introduction

Finite volume element methods (FVEMs) [1–3], which called box methods [4] in early times discretize the integral form of conservation law of differential equation by choosing linear or bilinear finite element space as trial space. The methods, which are also called generalized difference methods (GDMs) [5,6] in China have been widely used in numerical partial differential equations because they keep the conservation law of mass or energy. In recent years, some literature focused on the error estimates of linear or bilinear finite volume element methods; see the references [7–11]. Xu and Zou [12] developed an abstract framework to give a unified presentation of the finite volume methods and a unified study of the convergence theory of the finite volume method. Cai, Douglas and Park [13] constructed a high order finite volume element method by mixed variational principle. Shu, Yu and Huang [14] presented a symmetric finite volume element scheme on quadrilateral grids and Wang [15] presented an alternating direction finite volume element method by perturbing the differential equations. Plexousakis and Zouraris [16] derived a class of high order finite volume methods for solving one dimensional elliptic equations. Yang et al. [17,18] constructed and analyzed second order finite volume element schemes for two and three dimensional elliptic equations on quadrilateral meshes by the use of affine quadratic bases.

Essentially, both finite element and finite volume element are methods based on interpolations. By approximation theory, we know the numerical derivatives have only k-th order accuracy for interpolating polynomials of order k in general. But this fact does not exclude the possibility that the approximation of derivatives may be of higher order accuracy at some special points, which are called optimal stress points [19]. Based on optimal stress points of interpolation, the superconvergence

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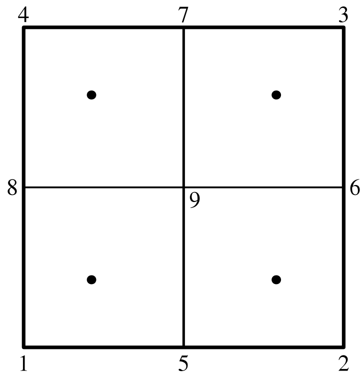

Fig. 1. A reference element and its four optimal stress points (•).

theory of finite element method has been studied intensively [19–21]. For finite volume element method, some articles and books discussed the superconvergence of numerical gradients at some points, especially for linear and cubic Hermite elements [5,7,14,16]. As is well known, finite volume element method uses a volume integral formulation of the differential equation with a finite partitioning set of volume to discretize the equation. For the elliptic equation, the key point of finite volume element method is to discretize the normal derivatives of the unknown function along the boundaries of control volumes. So if the partial derivatives have higher discretization accuracy, the finite volume element schemes may have higher order accuracy. Following this idea, by using optimal stress points as the vertices of control volumes, we can construct a superconvergent finite volume element scheme. In [22], Guo and Wang proposed a quadratic finite volume element method for two-point boundary value problems based on this idea and proved that the scheme has third order accuracy with respect to discrete H^1 seminorm. In this article, we systematically study the superconvergence of the biquadratic finite volume element method based on optimal stress points for two dimensional Possion's equations.

The remainder of the article is organized as follows. In Section 2, we briefly review the biquadratic interpolation and its four optimal stress points [19]. In Section 3, we construct superconvergent finite volume element scheme by using optimal stress points as the vertices of control volumes. In order to solve the scheme more efficiently, we write the biquadratic finite volume element scheme in tensor product form and solve it by alternating direction technique [23,24]. In Section 4, we obtain optimal H^1 semi-norm and L^2 norm error estimates. In Section 5, superconvergence results, including numerical gradients at optimal stress points and displacements at nodal points, are obtained. Finally, a numerical example is given in Section 6 to show that the method has truly high computational efficiency.

2. Review of biquadratic interpolation and its optimal stress points

In this section, we briefly state some results of biquadratic interpolation for our purpose. Further reading can be found in book [19]. Consider an element E_{ij} and a smooth function $u(x, y)$ defined on E_{ij}. Assume that $E_{ij} = [x_{2i-2}, x_{2i}] \times [y_{2j-2}, y_{2j}]$ is a rectangular element, which is illustrated in Fig. 1. Denote by $h_{i,x} = (x_{2i} - x_{2i-2})/2$, $h_{j,y} = (y_{2j} - y_{2j-2})/2$. Let $\xi = (x - x_{2i-1})/h_{i,x}$, $\eta = (y - y_{2j-1})/h_{j,y}$, $\{\xi_k\}^{\mathrm{T}} = \{-1, 1, 1, -1, 0, 1, 0, -1, 0\}$, $\{\eta_k\}^{\mathrm{T}} = \{-1, -1, 1, 1, -1, 0, 1, 0, 0\}$, where (x_{2i-1}, y_{2j-1}) is the center of element E_{ij}. Then the nine shape functions of biquadratic interpolation can be stated as follows

$$\psi_k(\xi, \eta) = \frac{1}{4}\xi_k\eta_k\xi\eta(1 + \xi_k\xi)(1 + \eta_k\eta), \quad k = 1, 2, 3, 4$$
$$\psi_k(\xi, \eta) = \frac{1}{2}\eta_k\eta(1 - \xi^2)(1 + \eta_k\eta), \quad k = 5, 7 \qquad \psi_k(\xi, \eta) = \frac{1}{2}\xi_k\xi(1 + \xi_k\xi)(1 - \eta^2), \quad k = 6, 8$$
$$\psi_9(\xi, \eta) = (1 - \xi^2)(1 - \eta^2).$$

Hence, the biquadratic interpolating polynomial of $u(x, y)$ over E_{ij} reads

$$\pi_2 u = \sum_{k=1}^{9} u_{2i-1+\xi_k, 2j-1+\eta_k}\psi_k(\xi, \eta). \tag{2.1}$$

Let

$$x_{o1,i} = x_{2i-1} - \frac{1}{\sqrt{3}}h_{i,x}, \qquad x_{o2,i} = x_{2i-1} + \frac{1}{\sqrt{3}}h_{i,x},$$
$$y_{o1,j} = y_{2j-1} - \frac{1}{\sqrt{3}}h_{j,y}, \qquad y_{o2,j} = y_{2j-1} + \frac{1}{\sqrt{3}}h_{j,y}.$$

A straightforward computation shows

$$\frac{\partial \pi_2 u}{\partial x}(x_{o1,i}, y_{o1,j}) = \frac{\partial u}{\partial x}(x_{o1,i}, y_{o1,j}) - \frac{\sqrt{3}}{108} h_{i,x}^3 \frac{\partial^4 u}{\partial x^4}(x_{o1,i}, y_{o1,j}) - \frac{\sqrt{3}}{27} h_{j,y}^3 \frac{\partial^4 u}{\partial x \partial y^3}(x_{o1,i}, y_{o1,j}) + O(h_{i,x}^4 + h_{j,y}^4), \tag{2.2}$$

$$\frac{\partial \pi_2 u}{\partial y}(x_{o1,i}, y_{o1,j}) = \frac{\partial u}{\partial y}(x_{o1,i}, y_{o1,j}) - \frac{\sqrt{3}}{27} h_{i,x}^3 \frac{\partial^4 u}{\partial x^3 \partial y}(x_{o1,i}, y_{o1,j}) - \frac{\sqrt{3}}{108} h_{j,y}^3 \frac{\partial^4 u}{\partial y^4}(x_{o1,i}, y_{o1,j}) + O(h_{i,x}^4 + h_{j,y}^4). \tag{2.3}$$

We know from (2.2) and (2.3) that the gradient of biquadratic interpolation $\pi_2 u$ at point $(x_{o1,i}, y_{o1,j})$ approximates the corresponding gradient of u with third order accuracy. In general, the approximating accuracy of gradient is only second order for biquadratic interpolation. We call point $(x_{o1,i}, y_{o1,j})$ an optimal stress point (OSP) of biquadratic interpolation [5,19]. Similar arguments can prove that points $(x_{o2,i}, y_{o1,j})$, $(x_{o2,i}, y_{o2,j})$ and $(x_{o1,i}, y_{o2,j})$ are other three optimal stress points of biquadratic interpolation. In the next section, we will use these four optimal stress points as vertices of a control volume to construct the superconvergent finite volume element scheme.

3. Biquadratic FVEM based on optimal stress points

Consider the following two dimensional Poisson's equation on domain $\Omega = (0, 1)^2$,

$$-\Delta u = f(x, y), \quad (x, y) \in \Omega, \qquad u|_{\partial\Omega} = 0, \tag{3.1}$$

where $f(x, y)$ is sufficiently smooth.

First, give a rectangular partition Q_h for domain Ω and the nodes are denoted by $(x_i, y_j), i(j) = 0, 1, \ldots, 2N_x(2N_y)$. Q_h has $N_x \times N_y$ elements, which are denoted by $E_{ij} = [x_{2i-2}, x_{2i}] \times [y_{2j-2}, y_{2j}], i(j) = 1, 2, \ldots, N_x(N_y)$. The point (x_{2i-1}, y_{2j-1}) is the center of E_{ij}. We also use the notations in Section 2. Assume that $\alpha_k(x)(k = 0, 1, \ldots, 2N_x)$ and $\beta_l(y)(l = 0, 1, \ldots, 2N_y)$ are piecewise quadratic interpolating base functions in x and y directions, respectively, then the piecewise biquadratic interpolation of $u(x, y)$ on Q_h reads

$$\pi_2 u = \sum_{k=0}^{2N_x} \sum_{l=0}^{2N_y} \alpha_k(x)\beta_l(y)u(x_k, y_l). \tag{3.2}$$

From Section 2, we know there are four optimal stress points for $\pi_2 u(x, y)$ in E_{ij}. Now we use these optimal stress points to construct a control volume for each nodal point. There are four different categories for all control volumes, which are

- For nodal point (x_{2i-1}, y_{2j-1}), its control volume is $V_{2i-1,2j-1} = [x_{o1,i}, x_{o2,i}] \times [y_{o1,j}, y_{o2,j}]$, where $i(j) = 1, 2, \ldots, N_x(N_y)$.
- For nodal point (x_{2i}, y_{2j-1}), its control volume is $V_{2i,2j-1} = [x_{o2,i}, x_{o1,i+1}] \times [y_{o1,j}, y_{o2,j}]$, where $i(j) = 1, 2, \ldots, N_x - 1(N_y)$.
- For nodal point (x_{2i-1}, y_{2j}), its control volume is $V_{2i-1,2j} = [x_{o1,i}, x_{o2,i}] \times [y_{o2,j}, y_{o1,j+1}]$, where $i, j = 1, 2, \ldots, N_x (N_y - 1)$.
- For nodal point (x_{2i}, y_{2j}), its control volume is $V_{2i,2j} = [x_{o2,i}, x_{o1,i+1}] \times [y_{o2,j}, y_{o1,j+1}]$, where $i(j) = 1, 2, \ldots, N_x - 1 (N_y - 1)$.

For boundary nodes, their control volumes should include the corresponding boundary points.

Next, integrate (3.1) over V_{ij}, then by Green's formula, the conservative integral form of (3.1) reads, finding $u \in H_0^1(\Omega)$, such that

$$-\int_{\partial V_{ij}} \frac{\partial u}{\partial \nu} \mathrm{d}s = \int_{V_{ij}} f(x, y)\mathrm{d}x\mathrm{d}y, \quad i(j) = 1, 2, \ldots, 2N_x - 1(2N_y - 1), \tag{3.3}$$

where ν denotes the unit outward normal vector of ∂V_{ij}, the boundary of V_{ij}. Suppose that $U_h \subset H_0^1(\Omega)$ is a finite element subspace over partition Q_h, which is spanned by $\{\alpha_k(x)\beta_l(y)\}, k(l) = 0, 1, \ldots, 2N_x(2N_y)$. In (3.3), replace u by $u_h \in U_h$, then the finite volume element scheme based on optimal stress points for (3.1) reads

$$-\int_{\partial V_{ij}} \frac{\partial u_h}{\partial \nu} \mathrm{d}s = \int_{V_{ij}} f(x, y)\mathrm{d}x\mathrm{d}y, \quad i(j) = 1, 2, \ldots, 2N_x - 1(2N_y - 1), \tag{3.4}$$

where in (3.4), $u_h = \sum_{k=0}^{2N_x}\sum_{l=0}^{2N_y}\alpha_k(x)\beta_l(y)u_{k,l}$ and $u_{k,l} = u_h(x_k, y_l)$. (3.4) can be written as tensor product form. Let

$$\begin{aligned}
&U = \left[u_{0,0}, u_{0,1}, \ldots, u_{0,2N_y}, u_{1,0}, u_{1,1}, \ldots, u_{1,2N_y}, \ldots, u_{2N_x,0}, u_{2N_x,1}, \ldots, u_{2N_x,2N_y}\right]^{\mathrm{T}},\\
&C_x = [cx_{i,k}]_{(2N_x-1)\times(2N_x+1)}, \qquad C_y = [cy_{j,l}]_{(2N_y-1)\times(2N_y+1)},\\
&A_x = [ax_{i,k}]_{(2N_x-1)\times(2N_x+1)}, \qquad A_y = [ay_{j,l}]_{(2N_y-1)\times(2N_y+1)},\\
&\Phi = \left[\phi_{1,1}, \phi_{1,2}, \ldots, \phi_{1,2N_y-1}, \phi_{2,1}, \phi_{2,2}, \ldots, \phi_{2,2N_y-1}, \cdots, \phi_{2N_x-1,1}, \ldots, \phi_{2N_x-1,2N_y-1}\right]^{\mathrm{T}},
\end{aligned}$$

where

$$\begin{aligned}
&cx_{2i-1,k} = \int_{x_{o1,i}}^{x_{o2,i}} \alpha_k(x)\mathrm{d}x, \quad i = 1, 2, \ldots, N_x,\ k = 0, 1, \ldots, 2N_x,\\
&cx_{2i,k} = \int_{x_{o2,i}}^{x_{o1,i+1}} \alpha_k(x)\mathrm{d}x, \quad i = 1, 2, \ldots, N_x - 1,\ k = 0, 1, \ldots, 2N_x,\\
&cy_{2j-1,l} = \int_{y_{o1,j}}^{y_{o2,j}} \beta_l(y)\mathrm{d}y, \quad j = 1, 2, \ldots, N_y,\ l = 0, 1, \ldots, 2N_y,\\
&cy_{2j,l} = \int_{y_{o2,j}}^{y_{o1,j+1}} \beta_l(y)\mathrm{d}y, \quad j = 1, 2, \ldots, N_y - 1,\ l = 0, 1, \ldots, 2N_y,\\
&ax_{2i-1,k} = \alpha_k'\left(x_{o1,i}\right) - \alpha_k'\left(x_{o2,i}\right), \quad i = 1, 2, \ldots, N_x,\ k = 0, 1, \ldots, 2N_x,\\
&ax_{2i,k} = \alpha_k'\left(x_{o2,i}\right) - \alpha_k'\left(x_{o1,i+1}\right), \quad i = 1, 2, \ldots, N_x - 1,\ k = 0, 1, \ldots, 2N_x,\\
&ay_{2j-1,l} = \beta_l'\left(y_{o1,j}\right) - \beta_l'\left(y_{o2,j}\right), \quad j = 1, 2, \ldots, N_y,\ l = 0, 1, \ldots, 2N_y,\\
&ay_{2j,l} = \beta_l'\left(y_{o2,j}\right) - \beta_l'\left(y_{o1,j+1}\right), \quad j = 1, 2, \ldots, N_y - 1,\ l = 0, 1, \ldots, 2N_y,\\
&\phi_{i,j} = \int_{V_{ij}} f(x, y)\mathrm{d}x\mathrm{d}y, \quad i = 1, 2, \ldots, 2N_x - 1,\ j = 1, 2, \ldots, 2N_y - 1.
\end{aligned}$$

Using the above notations, (3.4) can be rewritten as

$$\left(A_x \otimes C_y + C_x \otimes A_y\right) U = \Phi, \tag{3.5}$$

where $\otimes$ represents Kronecker tensor product, refer to [23] for its operations in detail.

In practical computation, Φ can be approximated as $\Phi \approx C_x \otimes C_y F$, where F is a vector analogous to U, but its entries are $f_{i,j} = f(x_i, y_j)$, $i(j) = 0, 1, \ldots, 2N_x(2N_y)$.

For biquadratic interpolation, a straightforward computation shows

$$cx_{2i-1,k} = \begin{cases} \dfrac{1}{9\sqrt{3}}h_{i,x}, & k = 2i-2,\ 2i,\\ \dfrac{16}{9\sqrt{3}}h_{i,x}, & k = 2i-1,\\ 0, & \text{otherwise} \end{cases} \qquad i = 1, 2, \ldots, N_x, \tag{3.6}$$

$$cx_{2i,k} = \begin{cases} -\dfrac{\sqrt{3}}{54}h_{i,x}, & k = 2i-2,\\ \dfrac{2(9-4\sqrt{3})}{27}h_{i,x}, & k = 2i-1,\\ \dfrac{18-\sqrt{3}}{54}(h_{i,x}+h_{i+1,x}), & k = 2i,\\ \dfrac{2(9-4\sqrt{3})}{27}h_{i+1,x}, & k = 2i+1,\\ -\dfrac{\sqrt{3}}{54}h_{i+1,x}, & k = 2i+2,\\ 0, & \text{otherwise} \end{cases} \qquad i = 1, 2, \ldots, N_x - 1, \tag{3.7}$$

$$ax_{2i-1,k} = \begin{cases} -\dfrac{1}{h_{i,x}}\dfrac{2}{\sqrt{3}}, & k = 2i-2,\ 2i,\\ \dfrac{1}{h_{i,x}}\dfrac{4}{\sqrt{3}}, & k = 2i-1,\\ 0, & \text{otherwise} \end{cases} \qquad i = 1, 2, \ldots, N_x, \tag{3.8}$$

$$ax_{2i,k} = \begin{cases} \dfrac{1}{h_{i,x}}\left(\dfrac{1}{\sqrt{3}} - \dfrac{1}{2}\right), & k = 2i-2, \\ -\dfrac{1}{h_{i,x}}\dfrac{2}{\sqrt{3}}, & k = 2i-1, \\ \left(\dfrac{1}{h_{i,x}} + \dfrac{1}{h_{i+1,x}}\right)\left(\dfrac{1}{\sqrt{3}} + \dfrac{1}{2}\right), & k = 2i, \\ -\dfrac{1}{h_{i+1,x}}\dfrac{2}{\sqrt{3}}, & k = 2i+1, \\ \dfrac{1}{h_{i+1,x}}\left(\dfrac{1}{\sqrt{3}} - \dfrac{1}{2}\right), & k = 2i+2, \\ 0, & \text{otherwise} \end{cases} \qquad i = 1, 2, \ldots, N_x - 1, \tag{3.9}$$

C_y and A_y are defined analogously to C_x and A_x, respectively.

(3.5) can be solved by alternating direction iteration method. Introducing iteration parameter τ and arbitrary U^0, then perturbing (3.5), we have

$$\left(C_x \otimes I + \frac{1}{2}\tau A_x \otimes I\right)\left(I \otimes C_y + \frac{1}{2}\tau I \otimes A_y\right)U^n = \left(C_x \otimes I - \frac{1}{2}\tau A_x \otimes I\right)\left(I \otimes C_y - \frac{1}{2}\tau I \otimes A_y\right)U^{n-1} + \tau\Phi. \tag{3.10}$$

Using the discretization idea based on Douglas ADI scheme in finite difference method and finite element method [23,24], we obtain the following alternating direction scheme

$$\left(C_x \otimes I + \frac{1}{2}\tau A_x \otimes I\right)\left(U^{n-\frac{1}{2}} - U^{n-1}\right) = -\tau\left(A_x \otimes C_y + C_x \otimes A_y\right)U^{n-1} + \tau\Phi, \tag{3.11}$$

$$\left(I \otimes C_y + \frac{1}{2}\tau I \otimes A_y\right)\left(U^n - U^{n-1}\right) = U^{n-\frac{1}{2}} - U^{n-1}. \tag{3.12}$$

(3.11) is solved in x direction, meanwhile, (3.12) is solved in y direction. From the theory of alternating direction method, we know the iteration procedure (3.11) with (3.12) is convergent when τ is suitably chosen.

4. H^1 norm and L^2 norm error estimates

We have derived a kind of biquadratic finite volume element method based on optimal stress points in Section 3. In this section, we further analyze the convergence of the scheme by using energy norm.

Denote $\chi_{i,j}$ by characteristic function over control volume $V_{i,j}$. Let

$$\Pi_h^*\varphi_h = \sum_{i=0}^{2N_x}\sum_{j=0}^{2N_y}\varphi_h(x_i, y_j)\chi_{i,j}, \quad \forall \varphi_h \in U_h,$$

$$a(u, \Pi_h^*\varphi_h) = -\sum_{i=0}^{2N_x}\sum_{j=0}^{2N_y}\varphi_h(x_i, y_j)\int_{\partial V_{i,j}}\frac{\partial u}{\partial \nu}\mathrm{d}s, \quad u \in H_0^1(\Omega),\ \varphi_h \in U_h,$$

$$(f, \Pi_h^*\varphi_h) = \sum_{i=0}^{2N_x}\sum_{j=0}^{2N_y}\varphi_h(x_i, y_j)\int_{V_{i,j}} f(x, y)\mathrm{d}x\mathrm{d}y, \quad \varphi_h \in U_h.$$

Noting that $\varphi_h|_{\partial\Omega} = 0$, the conservative integral form (3.3) is equivalent to

$$a(u, \Pi_h^*\varphi_h) = (f, \Pi_h^*\varphi_h), \quad \forall \varphi_h \in U_h. \tag{4.1}$$

Analogously, the finite volume element scheme (3.4) is equivalent to

$$a(u_h, \Pi_h^*\varphi_h) = (f, \Pi_h^*\varphi_h), \quad \forall \varphi_h \in U_h. \tag{4.2}$$

Suppose that Q_h is a quasi-uniformly regular partition; i.e., there exist constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$, satisfying

$$\alpha_1 \max_i h_{i,x} \le \min_i h_{i,x}, \qquad \alpha_2 \max_j h_{j,y} \le \min_j h_{j,y}, \qquad \alpha_3 h_{j,y} \le h_{i,x} \le \alpha_4 h_{j,y}.$$

Let $h = \max(\max h_{i,x}, \max h_{j,y})$. For arbitrary grid function $\{\varphi_{i,j}\}$ defined on element E_{ij}, we introduce the following notations

$$\delta_{\bar{x}}\varphi_{2i,2j} = \frac{1}{h_{i,x}}\left(\varphi_{2i,2j} - \varphi_{2i-1,2j}\right), \qquad \delta_{\bar{y}}\varphi_{2i,2j} = \frac{1}{h_{j,y}}\left(\varphi_{2i,2j} - \varphi_{2i,2j-1}\right).$$

We convert the integrals on the edges of a control volume to the related elements, then

$$a(u_h, \Pi_h^*\varphi_h) = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left[a_{i,j,x}(u_h, \Pi_h^*\varphi_h) + a_{i,j,y}(u_h, \Pi_h^*\varphi_h)\right], \tag{4.3}$$

where

$$a_{i,j,x}(u_h, \Pi_h^*\varphi_h) = \sum_{l=0}^{2}\sum_{k=0}^{1} h_{i,x}\delta_{\bar{x}}\varphi_{2i-k,2j-l}\int_{\beta_{1,l}}^{\beta_{2,l}} \frac{\partial u_h}{\partial x}\left(x_{2i-1} + (-1)^k\frac{h_{i,x}}{\sqrt{3}}, y\right)\mathrm{d}y, \tag{4.4}$$

$$a_{i,j,y}(u_h, \Pi_h^*\varphi_h) = \sum_{k=0}^{2}\sum_{l=0}^{1} h_{j,y}\delta_{\bar{y}}\varphi_{2i-k,2j-l}\int_{\alpha_{k,1}}^{\alpha_{k,2}} \frac{\partial u_h}{\partial y}\left(x, y_{2j-1} + (-1)^l\frac{h_{j,y}}{\sqrt{3}}\right)\mathrm{d}x. \tag{4.5}$$

We note that in (4.4) and (4.5),

$$\beta_{1,l} = \begin{cases} y_{2j-1} + \frac{1}{\sqrt{3}}h_{j,y}, & l = 0,\\ y_{2j-1} - \frac{1}{\sqrt{3}}h_{j,y}, & l = 1,\\ y_{2j-2}, & l = 2, \end{cases} \qquad \beta_{2,l} = \begin{cases} y_{2j}, & l = 0,\\ y_{2j-1} + \frac{1}{\sqrt{3}}h_{j,y}, & l = 1,\\ y_{2j-1} - \frac{1}{\sqrt{3}}h_{j,y}, & l = 2, \end{cases}$$

$$\alpha_{k,1} = \begin{cases} x_{2i-1} + \frac{1}{\sqrt{3}}h_{i,x}, & k = 0,\\ x_{2i-1} - \frac{1}{\sqrt{3}}h_{i,x}, & k = 1,\\ x_{2i-2}, & k = 2, \end{cases} \qquad \alpha_{k,2} = \begin{cases} x_{2i}, & k = 0,\\ x_{2i-1} + \frac{1}{\sqrt{3}}h_{i,x}, & k = 1,\\ x_{2i-1} - \frac{1}{\sqrt{3}}h_{i,x}, & k = 2. \end{cases}$$

Denote $\|\cdot\|_s$ and $|\cdot|_s$ by continuous norm and continuous semi-norm of order s in Sobolev space, respectively. Define discrete H^1 semi-norm and discrete L^2 norm in space U_h respectively by

$$|u_h|_{1,h}^2 = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(|u_h|_{1,h,i,j,x}^2 + |u_h|_{1,h,i,j,y}^2\right), \qquad \|u_h\|_{0,h}^2 = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\|u_h\|_{0,h,i,j}^2, \quad \forall\, u_h \in U_h, \tag{4.6}$$

where

$$|u_h|_{1,h,i,j,x}^2 = \frac{h_{i,x}h_{j,y}}{3}\left\{\sum_{l=0,2}\sum_{k=0}^{1}\left(\delta_{\bar{x}}u_{2i-k,2j-l}\right)^2 + 4\sum_{k=0}^{1}\left(\delta_{\bar{x}}u_{2i-k,2j-1}\right)^2\right\},$$

$$|u_h|_{1,h,i,j,y}^2 = \frac{h_{i,x}h_{j,y}}{3}\left\{\sum_{k=0,2}\sum_{l=0}^{1}\left(\delta_{\bar{y}}u_{2i-k,2j-l}\right)^2 + 4\sum_{l=0}^{1}\left(\delta_{\bar{y}}u_{2i-1,2j-l}\right)^2\right\},$$

$$\|u_h\|_{0,h,i,j}^2 = \frac{h_{i,x}h_{j,y}}{9}\left\{\sum_{k=0,2}\left(\sum_{l=0,2}u_{2i-k,2j-l}^2 + 4u_{2i-k,2j-1}^2\right) + 4\sum_{l=0,2}u_{2i-1,2j-l}^2 + 16u_{2i-1,2j-1}^2\right\}.$$

Lemma 1. *For $\forall\, u_h \in U_h$, $|u_h|_{1,h}$ is equivalent to $|u_h|_1$ and $\|u_h\|_{0,h}$ is equivalent to $\|u_h\|_0$, that is, the following inequalities hold*

$$\sqrt{\frac{2}{5}}|u_h|_{1,h} \le |u_h|_1 \le \frac{2}{\sqrt{3}}|u_h|_{1,h}, \tag{4.7}$$
$$\frac{2}{5}\|u_h\|_{0,h} \le \|u_h\|_0 \le \|u_h\|_{0,h}.$$

Proof. From the definition of $|u_h|_1$, we know

$$|u_h|_1^2 = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}(|u_h|_{1,i,j,x}^2 + |u_h|_{1,i,j,y}^2).$$

Take transforms $\xi = (x - x_{2i-1})/h_{i,x}$, $\eta = (y - y_{2j-1})/h_{j,y}$, then

$$|u_h|_{1,i,j,x}^2 = \frac{h_{j,y}}{h_{i,x}}\int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial u_h}{\partial \xi}\right)^2\mathrm{d}\xi\,\mathrm{d}\eta.$$

Let

$$D_x u = [u_{2i,2j-2} - u_{2i-1,2j-2}, u_{2i-1,2j-2} - u_{2i-2,2j-2}, 2(u_{2i,2j-1} - u_{2i-1,2j-1}), \\ 2(u_{2i-1,2j-1} - u_{2i-2,2j-1}), u_{2i,2j} - u_{2i-1,2j}, u_{2i-1,2j} - u_{2i-2,2j}]^{\mathrm{T}}.$$

Noting that u_h is a biquadratic interpolation on $E_{ij} = [x_{2i-2}, x_{2i}] \times [y_{2j-2}, y_{2j}]$, a straightforward computation shows

$$|u_h|^2_{1,i,j,x} = \frac{h_{j,y}}{h_{i,x}} (D_x u)^{\mathrm{T}} M D_x u,$$

where M is a matrix,

$$M = \frac{1}{90} \begin{bmatrix} 28 & -4 & 7 & -1 & -7 & 1 \\ -4 & 28 & -1 & 7 & 1 & -7 \\ 7 & -1 & 28 & -4 & 7 & -1 \\ -1 & 7 & -4 & 28 & -1 & 7 \\ -7 & 1 & 7 & -1 & 28 & -4 \\ 1 & -7 & -1 & 7 & -4 & 28 \end{bmatrix}.$$

The eigenvalues of matrix M are $\lambda_l = \frac{4}{9}, \frac{4}{9}, \frac{1}{3}, \frac{1}{3}, \frac{8}{45}, \frac{2}{15}$, from which we can obtain

$$\frac{2}{15} \frac{h_{j,y}}{h_{i,x}} (D_x u)^{\mathrm{T}} D_x u \le |u_h|^2_{1,i,j,x} \le \frac{4}{9} \frac{h_{j,y}}{h_{i,x}} (D_x u)^{\mathrm{T}} D_x u.$$

From the definition of $|u_h|_{1,h,i,j,x}$, we obtain

$$\frac{2}{5} |u_h|^2_{1,h,i,j,x} \le |u_h|^2_{1,i,j,x} \le \frac{4}{3} |u_h|^2_{1,h,i,j,x}.$$

Analogously, we have

$$\frac{2}{5} |u_h|^2_{1,h,i,j,y} \le |u_h|^2_{1,i,j,y} \le \frac{4}{3} |u_h|^2_{1,h,i,j,y}.$$

Adding the above two inequalities, we get (4.7). The another inequality of Lemma 1 can be proved analogously. Lemma 1 is proved. □

Lemma 2.

$$a(u_h, \Pi_h^* u_h) \ge 0.449927 |u_h|^2_{1,h} \ge 0.337445 |u_h|^2_1, \quad \forall u_h \in U_h. \tag{4.8}$$

Proof. By (4.3), further computing the integrals in (4.4), we have

$$a_{i,j,x}(u_h, \Pi_h^* u_h) = \frac{h_{j,y}}{h_{i,x}} (D_x u)^{\mathrm{T}} M D_x u,$$

where matrix M is defined as

$$M = \frac{1}{108} \begin{bmatrix} 16+11\sqrt{3} & 20-13\sqrt{3} & 2+4\sqrt{3} & 34-20\sqrt{3} & -(2+\sqrt{3}) & 2-\sqrt{3} \\ 20-13\sqrt{3} & 16+11\sqrt{3} & 34-20\sqrt{3} & 2+4\sqrt{3} & 2-\sqrt{3} & -(2+\sqrt{3}) \\ 2+\sqrt{3} & -2+\sqrt{3} & 8(2+\sqrt{3}) & 8(-2+\sqrt{3}) & 2+\sqrt{3} & -2+\sqrt{3} \\ -2+\sqrt{3} & 2+\sqrt{3} & 8(-2+\sqrt{3}) & 8(2+\sqrt{3}) & -2+\sqrt{3} & 2+\sqrt{3} \\ -(2+\sqrt{3}) & 2-\sqrt{3} & 2+4\sqrt{3} & 34-20\sqrt{3} & 16+11\sqrt{3} & 20-13\sqrt{3} \\ 2-\sqrt{3} & -(2+\sqrt{3}) & 34-20\sqrt{3} & 2+4\sqrt{3} & 20-13\sqrt{3} & 16+11\sqrt{3} \end{bmatrix}.$$

By Schur Decomposition, we have $M = QBQ^{\mathrm{T}}$, where Q is a 6×6 orthogonal matrix and B is an upper triangular matrix, which reads

$$B = \begin{bmatrix} 0.3849 & 0 & 0 & -0.0364633 & 0.0138133 & 0 \\ 0 & 0.333333 & -0.0315781 & 0 & 0 & -0.0119627 \\ 0 & 0 & 0.19245 & 0 & 0 & -0.0533706 \\ 0 & 0 & 0 & 0.222222 & 0.0616271 & 0 \\ 0 & 0 & 0 & 0 & 0.3849 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.333333 \end{bmatrix}.$$

Let $v = Q^{\mathrm{T}} D_x u$, then by Cauchy inequality, we can obtain $v^{\mathrm{T}} B v \ge 0.149976 v^{\mathrm{T}} v$ and

$$a_{i,j,x}(u_h, \Pi_h^* u_h) \ge 0.149976 \frac{h_{j,y}}{h_{i,x}} (D_x u)^{\mathrm{T}} D_x u = 0.449927 |u_h|^2_{1,h,i,j,x}.$$

Analogously, we have

$$a_{i,j,y}(u_h, \Pi_h^* u_h) \geq 0.449927|u_h|^2_{1,h,i,j,y}.$$

Adding the above two inequalities and by Lemma 1, we get (4.8). The proof is completed. $\square$

Lemma 3. *Suppose that Q_h is a quasi-uniform partition. Further assume that $u \in H_0^1(\Omega) \cap H^4(\Omega)$, then there exists a positive constant C, independent of mesh size h, such that*

$$|a(u - \pi_2 u, \Pi_h^* \varphi_h)| \leq Ch^3 |u|_4 |\varphi_h|_1. \tag{4.9}$$

In addition, if $u \in H_0^1(\Omega) \cap H^3(\Omega)$ only, then

$$|a(u - \pi_2 u, \Pi_h^* \varphi_h)| \leq Ch^2 |u|_3 |\varphi_h|_1. \tag{4.10}$$

Proof. From (4.3), we have

$$a(u - \pi_2 u, \Pi_h^* \varphi_h) = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left[a_{i,j,x}(u - \pi_2 u, \Pi_h^* \varphi_h) + a_{i,j,y}(u - \pi_2 u, \Pi_h^* \varphi_h) \right],$$

where $a_{i,j,x}(u - \pi_2 u, \Pi_h^* \varphi_h)$ and $a_{i,j,y}(u - \pi_2 u, \Pi_h^* \varphi_h)$ can be derived from (4.4) and (4.5), respectively. Here we only estimate one such integral, that is

$$\begin{aligned}
&\int_{y_{2j-2}}^{y_{2j-1} - \frac{1}{\sqrt{3}} h_{j,y}} \frac{\partial (u - \pi_2 u)}{\partial x} \left(x_{2i-1} - \frac{1}{\sqrt{3}} h_{i,x}, y \right) \mathrm{d}y = \frac{h_{j,y}}{h_{i,x}} \left\{ \int_{-1}^{-\frac{1}{\sqrt{3}}} \frac{\partial u}{\partial \xi} \left(-\frac{1}{\sqrt{3}}, \eta \right) \mathrm{d}\eta \right. \\
&\quad - \frac{1}{108} \Big[-(16 + 11\sqrt{3}) u(-1,-1) - 4(1 + 2\sqrt{3}) u(-1,0) + (2 + \sqrt{3}) u(-1,1) \\
&\quad + 4(-1 + 6\sqrt{3}) u(0,-1) + 16(-4 + 3\sqrt{3}) u(0,0) - 4u(0,1) \\
&\quad \left. + \ (20 - 13\sqrt{3}) u(1,-1) + (68 - 40\sqrt{3}) u(1,0) + (2 - \sqrt{3}) u(1,1) \Big] \right\} \\
&\triangleq \frac{h_{j,y}}{h_{i,x}} I(u).
\end{aligned}$$

$I(u)$ is a linear functional of $u \in H^4(E)$, where $E = [-1, 1]^2$ is a reference element. From the above formula, we have $|I(u)| \leq C\|u\|_{1,\infty,E}$. In addition, $H^4(E) \hookrightarrow C^1(E)$. Hence, $|I(u)| \leq C\|u\|_{4,E}$. A straightforward calculation shows $I(u) \equiv 0$ for $u = \xi^k \eta^l$, where $k, l = 0, 1, 2, 3$ and $k + l \leq 3$. By Bramble–Hilbert Lemma [25], we know $|I(u)| \leq C|u|_{4,E}$. By an integral transformation, we have $|I(u)| \leq Ch^3 |u|_{4,E_{ij}}$. The other integrals in $a_{i,j,x}(u - \pi_2 u, \Pi_h^* \varphi_h)$ have similar estimates. By using Cauchy inequality, we have

$$|a_{i,j,x}(u - \pi_2 u, \Pi_h^* \varphi_h)| \leq Ch^3 |u|_{4,E_{ij}} |\varphi_h|_{1,h,i,j,x}.$$

Analogously, we have

$$|a_{i,j,y}(u - \pi_2 u, \Pi_h^* \varphi_h)| \leq Ch^3 |u|_{4,E_{ij}} |\varphi_h|_{1,h,i,j,y}.$$

By using Cauchy inequality again and Lemma 1, we obtain

$$|a(u - \pi_2 u, \Pi_h^* \varphi_h)| \leq Ch^3 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} |u|_{4,E_{ij}} \left(|\varphi_h|_{1,h,i,j,x} + |\varphi_h|_{1,h,i,j,y} \right) \leq Ch^3 |u|_4 |\varphi_h|_1.$$

If $u \in H^3(\Omega)$ only, because $H^3(E) \hookrightarrow C^1(E)$, (4.10) also holds by the same arguments as above. $\square$

We note that (4.9) is a very important estimate for superconvergence analysis of numerical gradients at optimal stress points.

Theorem 1. *Assume that u is the solution to (3.1) and u_h is the solution of biquadratic finite volume element scheme (3.4). Let $\pi_2 u$ be the biquadratic interpolation projection of u onto trial space U_h. If $u \in H_0^1(\Omega) \cap H^4(\Omega)$, then we have*

$$|u_h - \pi_2 u|_1 \leq Ch^3 |u|_4. \tag{4.11}$$

If $u \in H_0^1(\Omega) \cap H^3(\Omega)$ only, then

$$|u_h - \pi_2 u|_1 \leq Ch^2 |u|_3. \tag{4.12}$$

Proof. From (4.1) and (4.2), we have

$$a(u-u_h, \Pi_h^*\varphi_h) = 0, \quad \forall \varphi_h \in U_h. \tag{4.13}$$

Using (4.8) and (4.13) we find that

$$|u_h - \pi_2 u|_1^2 \le 2.96344 a(u_h - \pi_2 u, \Pi_h^*(u_h - \pi_2 u)) = 2.96344 a(u - \pi_2 u, \Pi_h^*(u_h - \pi_2 u)).$$

This gives

$$|u_h - \pi_2 u|_1 \le 2.96344 \sup_{\varphi_h \in U_h} \frac{|a(u-\pi_2 u, \Pi_h^*\varphi_h)|}{|\varphi_h|_1}.$$

By Lemma 3, (4.11) and (4.12) can be easily proved. □

Theorem 2. *Assume that $u \in H_0^1(\Omega) \cap H^3(\Omega)$ is the solution to (3.1) and u_h is the solution of biquadratic finite volume element scheme (3.4), then there exits a positive constant C independent of mesh-size h, such that the following optimal H^1 seminorm error estimate holds*

$$|u - u_h|_1 \le Ch^2 |u|_3. \tag{4.14}$$

Proof. By the interpolation theory in Sobolev space [25], we have

$$|u - \pi_2 u|_1 \le Ch^2 |u|_3. \tag{4.15}$$

By (4.12) and (4.15), we can get (4.14). The proof is completed. □

Using the fact that H^1 norm and H^1 seminorm are equivalent in $H_0^1(\Omega)$, from (4.11) we can prove the following theorem.

Theorem 3. *Assume that $u \in H_0^1(\Omega) \cap H^4(\Omega)$ is the solution to (3.1) and u_h is the solution of biquadratic finite volume element scheme (3.4), then there exists a positive constant C independent of mesh size h, such that*

$$\|u - u_h\|_0 \le Ch^3 \|u\|_4. \tag{4.16}$$

As for the alternating direction scheme (3.11) with (3.12), we can regard the scheme as an approximation of a parabolic equation. In this situation, the iteration parameter τ is the time step and U^n tends to a steady solution (that is exactly the solution U of scheme (3.4)) when $n \to \infty$. Hence, we could get a reasonable solution from scheme (3.11) with (3.12).

5. Superconvergence results

In the first part of this section, we consider the superconvergence of numerical gradients at optimal stress points. We use the notations in Section 2, which are repeated as follows. $E_{ij} = [x_{2i-2}, x_{2i}] \times [y_{2j-2}, y_{2j}]$ is a rectangular element. $(x_{ok,i}, y_{ol,j})$ $(k, l = 1, 2)$ are four optimal stress points of biquadratic interpolation $\pi_2 u$ over E_{ij}. In Section 2, we have proved $\nabla(u - \pi_2 u)(x_{ok,i}, y_{ol,j}) = O(h_{i,x}^3 + h_{j,y}^3)$ for sufficiently smooth function u on E_{ij}. The conclusion will be restated from another point of view in the follow lemma.

Lemma 4 (*[19]*). *Suppose that Q_h is a quasi-uniform partition. Further assume that $u \in H^4(\Omega)$, then there exists a positive constant C independent of mesh size h, such that*

$$\left[\frac{1}{4N_xN_y}\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\sum_{k=1}^{2}\sum_{l=1}^{2}|\nabla(u-\pi_2 u)(x_{ok,i}, y_{ol,j})|^2\right]^{\frac{1}{2}} \le Ch^3|u|_4. \tag{5.1}$$

Theorem 4. *Suppose that Q_h is a quasi-uniform partition. Assume that $u \in H_0^1(\Omega) \cap H^4(\Omega)$ is the solution to (3.1) and u_h is the solution of biquadratic finite volume element scheme (3.4), then the following gradient superconvergence result at optimal stress points holds*

$$\left[\frac{1}{4N_xN_y}\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\sum_{k=1}^{2}\sum_{l=1}^{2}|\nabla(u-u_h)(x_{ok,i}, y_{ol,j})|^2\right]^{\frac{1}{2}} \le Ch^3|u|_4. \tag{5.2}$$

Proof. For E_{ij}, inverse property of finite element space implies

$$|\nabla(u_h - \pi_2 u)(x_{ok,i}, y_{ol,j})| \le Ch^{-1}|u_h - \pi_2 u|_{1,E_{ij}}, \tag{5.3}$$

from which we can obtain

$$\left[\frac{1}{4N_xN_y}\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\sum_{k=1}^{2}\sum_{l=1}^{2}|\nabla(u_h-\pi_2 u)(x_{ok,i},y_{ol,j})|^2\right]^{\frac{1}{2}} \le Ch^3|u|_4. \tag{5.4}$$

Combining (5.1) with (5.4), we easily get (5.2). Theorem 4 is proved. □

In the following, we consider the superconvergence of the nodal points. For arbitrary grid function $\{\varphi_{i,j}\}$ defined on element E_{ij}, we further introduce the following notations

$$\begin{aligned}
\delta_x\varphi_{2i-1,2j} &= \frac{1}{2h_{i,x}}\left(\varphi_{2i,2j}-\varphi_{2i-2,2j}\right), \qquad \delta_y\varphi_{2i,2j-1} = \frac{1}{2h_{j,y}}\left(\varphi_{2i,2j}-\varphi_{2i,2j-2}\right),\\
\delta_x^2\varphi_{2i-1,2j} &= \frac{1}{h_{i,x}^2}\left(\varphi_{2i,2j}-2\varphi_{2i-1,2j}+\varphi_{2i-2,2j}\right),\\
\delta_y^2\varphi_{2i,2j-1} &= \frac{1}{h_{j,y}^2}\left(\varphi_{2i,2j}-2\varphi_{2i,2j-1}+\varphi_{2i,2j-2}\right),\\
\delta_x\delta_y\varphi_{2i-1,2j-1} &= \frac{1}{4h_{i,x}h_{j,y}}\left(\varphi_{2i,2j}-\varphi_{2i-2,2j}-\varphi_{2i,2j-2}+\varphi_{2i-2,2j-2}\right),\\
\delta_{\bar{x}}\delta_{\bar{y}}\varphi_{2i,2j} &= \frac{1}{h_{i,x}h_{j,y}}\left(\varphi_{2i,2j}-\varphi_{2i-1,2j}-\varphi_{2i,2j-1}+\varphi_{2i-1,2j-1}\right).
\end{aligned} \tag{5.5}$$

Define discrete H^2 semi-norm by

$$|\varphi|_{2,h}^2 = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(|\varphi|_{2,h,i,j,x}^2+|\varphi|_{2,h,i,j,y}^2+|\varphi|_{2,h,i,j,x,y}^2\right), \tag{5.6}$$

where

$$\begin{aligned}
|\varphi|_{2,h,i,j,x}^2 &= \frac{2}{3}h_{i,x}h_{j,y}\left[(\delta_x^2\varphi_{2i-1,2j-2})^2+4(\delta_x^2\varphi_{2i-1,2j-1})^2+(\delta_x^2\varphi_{2i-1,2j})^2\right],\\
|\varphi|_{2,h,i,j,y}^2 &= \frac{2}{3}h_{i,x}h_{j,y}\left[(\delta_y^2\varphi_{2i-2,2j-1})^2+4(\delta_y^2\varphi_{2i-1,2j-1})^2+(\delta_y^2\varphi_{2i,2j-1})^2\right],\\
|\varphi|_{2,h,i,j,x,y}^2 &= h_{i,x}h_{j,y}\left[(\delta_{\bar{x}}\delta_{\bar{y}}\varphi_{2i,2j})^2+(\delta_{\bar{x}}\delta_{\bar{y}}\varphi_{2i-1,2j})^2+(\delta_{\bar{x}}\delta_{\bar{y}}\varphi_{2i,2j-1})^2+(\delta_{\bar{x}}\delta_{\bar{y}}\varphi_{2i-1,2j-1})^2\right].
\end{aligned}$$

Lemma 5. *Assume that $\varphi \in H^2(\Omega)$, then there exists $C > 0$, independent of mesh-size h, such that*

$$|\varphi|_{2,h} \le C|\varphi|_2. \tag{5.7}$$

Proof. Denote $\pi_2\varphi$ by the piecewise biquadratic interpolation of $\varphi(x,y)$ on the quasi-uniformly regular partition Q_h. First, a straightforward computation shows

$$\frac{2}{5}|\varphi|_{2,h}^2 \le |\pi_2\varphi|_2^2 \le |\varphi|_{2,h}^2. \tag{5.8}$$

Second, using a result in [19] (P167, Lemma 1), that is

$$|\pi_2\varphi|_2 \le C|\varphi|_2, \tag{5.9}$$

we can prove Lemma 5. □

Lemma 6. *Assume that $u \in H_0^1(\Omega)\cap H^5(\Omega)$ and $\varphi_h \in U_h$, then there exists a positive constant C, independent of mesh size h, such that*

$$|a(u-\pi_2 u, \pi_h^*\varphi_h)| \le Ch^4|\varphi_h|_{2,h}|u|_{4,h} + Ch^4|\varphi_h|_{1,h}|u|_{5,h}, \tag{5.10}$$

where $|u|_{4,h}^2$ includes the terms such as $\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{\partial^4 u}{\partial x^4}(x_{2i-1},y_{2j-1})^2 4h_{i,x}h_{j,y}$ and $|u|_{4,h}\to|u|_4$, $|u|_{5,h}\to|u|_5$ as $h\to 0$.

Proof. Noting that (4.3) and (4.4), by complicated computations using Taylor's expansion, we have

$$\begin{aligned} a_{i,j,x}(u-\pi_2 u, \pi_h^*\varphi_h) = & -\frac{2}{27}h_{i,x}h_{j,y}^5\delta_x\delta_y\varphi_{2i-1,2j-1}\frac{\partial^4 u}{\partial x\partial y^3}(x_{2i-1}, y_{2j-1}) - \frac{\sqrt{3}-1}{108}h_{i,x}^5 h_{j,y}\delta_x^2\varphi_{2i-1,2j-2}\frac{\partial^4 u}{\partial x^4}(x_{2i-1}, y_{2j-2}) \\ & - \frac{1}{54}h_{i,x}^5 h_{j,y}\delta_x^2\varphi_{2i-1,2j-1}\frac{\partial^4 u}{\partial x^4}(x_{2i-1}, y_{2j-1}) \\ & - \frac{\sqrt{3}-1}{108}h_{i,x}^5 h_{j,y}\delta_x^2\varphi_{2i-1,2j}\frac{\partial^4 u}{\partial x^4}(x_{2i-1}, y_{2j}) + r_{i,j,x}(u,\varphi_h), \end{aligned} \tag{5.11}$$

where $r_{i,j,x}(u, \varphi_h)$ is a complicated higher order remainder, which includes the terms such as

$$(\varphi_{2i-1,2j-2} - \varphi_{2i-2,2j-2})\sum_{\kappa_1+\kappa_2=5} c_{\kappa_1,\kappa_2} h_{i,x}^{\gamma_1} h_{j,y}^{\gamma_2}\frac{\partial^5 u}{\partial x^{\kappa_1}\partial y^{\kappa_2}}(x_{\kappa_1}, y_{\kappa_2}), \quad \gamma_1+\gamma_2 = 5 \quad \text{and} \quad \gamma_2 \ge 1.$$

Noting that

$$\delta_x\delta_y\varphi_{2i-1,2j-1} = \frac{1}{4}\left[\delta_{\bar{x}}\delta_{\bar{y}}\varphi_{2i,2j} + \delta_{\bar{x}}\delta_{\bar{y}}\varphi_{2i-1,2j} + \delta_{\bar{x}}\delta_{\bar{y}}\varphi_{2i,2j-1} + \delta_{\bar{x}}\delta_{\bar{y}}\varphi_{2i-1,2j-1}\right],$$

we have

$$(\delta_x\delta_y\varphi_{2i-1,2j-1})^2 \le \frac{1}{4h_{i,x}h_{j,y}}|\varphi_h|_{2,h,i,j,x,y}^2.$$

By Cauchy inequality for (5.11), we have

$$|a_{i,j,x}(u-\pi_2 u, \pi_h^*\varphi_h)| \le Ch^4(|\varphi_h|_{2,h,i,j,x}^2 + |\varphi_h|_{2,h,i,j,x,y}^2)^{\frac{1}{2}}|u|_{4,h,i,j} + Ch^4|\varphi_h|_{1,h,i,j,x}|u|_{5,h,i,j}. \tag{5.12}$$

$a_{i,j,y}(u-\pi_2 u, \pi_h^*\varphi_h)$ has a similar result as (5.12). Hence, we can prove Lemma 6. □

For $u, \varphi \in H^1(\Omega)$, denote by

$$b_{i,j,x}(u,\varphi) = \int_{E_{ij}}\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x}\mathrm{d}x\mathrm{d}y, \qquad b_{i,j,y}(u,\varphi) = \int_{E_{ij}}\frac{\partial u}{\partial y}\frac{\partial \varphi}{\partial y}\mathrm{d}x\mathrm{d}y$$

and define bilinear and linear functionals respectively by

$$b(u,\varphi) = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}(b_{i,j,x}(u,\varphi) + b_{i,j,y}(u,\varphi)), \quad (u,\varphi) = \int_\Omega u\varphi\,\mathrm{d}x\mathrm{d}y. \tag{5.13}$$

Lemma 7. *Suppose that Q_h is a quasi-uniform partition. For $u_h, \varphi_h \in U_h$, there exists a positive constant C, independent of mesh size h, such that*

$$|b(u_h,\varphi_h) - a(u_h, \pi_h^*\varphi_h)| \le Ch|u_h|_{1,h}|\varphi_h|_{2,h}. \tag{5.14}$$

Proof. From (5.13) and (4.3) we have

$$b(u_h,\varphi_h) - a(u_h,\pi_h^*\varphi_h) = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left[b_{i,j,x}(u_h,\varphi_h) - a_{i,j,x}(u_h,\pi_h^*\varphi_h) + b_{i,j,y}(u_h,\varphi_h) - a_{i,j,y}(u_h,\pi_h^*\varphi_h)\right]. \tag{5.15}$$

A straightforward computation shows

$$b_{i,j,x}(u_h,\varphi_h) - a_{i,j,x}(u_h,\pi_h^*\varphi_h) = T_1 + T_2, \tag{5.16}$$

where

$$T_1 = \frac{h_{i,x}h_{j,y}^3}{135}\left[(-18+5\sqrt{3})(\delta_x u_{2i-1,2j} + \delta_x u_{2i-1,2j-2}) + 16(-9+5\sqrt{3})\delta_x u_{2i-1,2j-1}\right]\cdot\delta_x\delta_y^2\varphi_{2i-1,2j-1},$$

$$T_2 = \frac{h_{i,x}^3 h_{j,y}}{270}\mathbf{v}^{\mathrm{T}}M\mathbf{w}.$$

In T_2, vectors $\mathbf{v}$, $\mathbf{w}$ and matrix M are defined sequentially by

$$\mathbf{v} = \begin{bmatrix}\delta_x^2 u_{2i-1,2j-2}\\ 2\delta_x^2 u_{2i-1,2j-1}\\ \delta_x^2 u_{2i-1,2j}\end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix}\delta_x^2\varphi_{2i-1,2j-2}\\ 2\delta_x^2\varphi_{2i-1,2j-1}\\ \delta_x^2\varphi_{2i-1,2j}\end{bmatrix}, \qquad M = \begin{bmatrix}53-30\sqrt{3} & 7 & -7\\ 52-30\sqrt{3} & 8 & 52-30\sqrt{3}\\ -7 & 7 & 53-30\sqrt{3}\end{bmatrix}.$$

By the equality

$$h_{j,y}\delta_x\delta_y^2\varphi_{2i-1,2j-1} = \frac{1}{2}\left[\delta_{\bar{x}}\delta_{\bar{y}}\varphi_{2i,2j} + \delta_{\bar{x}}\delta_{\bar{y}}\varphi_{2i-1,2j} - \delta_{\bar{x}}\delta_{\bar{y}}\varphi_{2i,2j-1} - \delta_{\bar{x}}\delta_{\bar{y}}\varphi_{2i-1,2j-1}\right],$$

we deduce that

$$(h_{j,y}\delta_x\delta_y^2\varphi_{2i-1,2j-1})^2 \le \frac{1}{h_{i,x}h_{j,y}}|\varphi_h|^2_{2,h,i,j,x,y}.$$

Hence, for T_1, by Cauchy inequality, we have

$$|T_1| \le \frac{18\sqrt{2}-5\sqrt{6}}{90}h_{j,y}|u_h|_{1,h,i,j,x}|\varphi_h|_{2,h,i,j,x,y}. \tag{5.17}$$

For T_2, also by Cauchy inequality, we have

$$|T_2| \le \frac{\sqrt{3}}{18}h_{i,x}^2|u_h|_{2,h,i,j,x}|\varphi_h|_{2,h,i,j,x}. \tag{5.18}$$

From (5.17) and (5.18), we can get the estimate of $b_{i,j,x}(u_h,\varphi_h) - a_{i,j,x}(u_h,\pi_h^*\varphi_h)$. Similar results hold for $b_{i,j,y}(u_h,\varphi_h) - a_{i,j,y}(u_h,\pi_h^*\varphi_h)$. Substituting these estimates into (5.15), by Cauchy inequality and inverse estimate, we have

$$|b(u_h,\varphi_h) - a(u_h,\pi_h^*\varphi_h)| \le C_1h|u_h|_{1,h}|\varphi_h|_{2,h} + C_2h^2|u_h|_{2,h}|\varphi_h|_{2,h} \le Ch|u_h|_{1,h}|\varphi_h|_{2,h}.$$

Lemma 7 is proved. □

Theorem 5. *Suppose that Q_h is a quasi-uniform partition. Assume that $u \in H_0^1(\Omega)\cap H^5(\Omega)$ is the solution to (3.1), π_2u is the piecewise biquadratic interpolation and u_h is the solution of biquadratic finite volume element scheme (3.4), then the following nodal point superconvergence result holds*

$$\|u_h - \pi_2u\|_0 \le Ch^4(|u|_4 + |u|_{4,h} + |u|_{5,h}), \qquad \lim_{h\to 0}|u|_{l,h} = |u|_l,\ l = 4, 5. \tag{5.19}$$

Proof. We use a modified dual argument technique ([26], p. 127) to prove this theorem. Let us introduce an auxiliary problem, that is, for given $g_h = u_h - \pi_2u$, finding $\varphi \in H_0^1(\Omega)$, such that

$$b(v,\varphi) = (g_h, v), \quad \forall\, v \in H_0^1(\Omega). \tag{5.20}$$

By the differential equation theory there exists a unique solution to problem (5.20), satisfying $\|\varphi\|_2 \le C\|g_h\|_0$. It follows from (5.20) that

$$\|g_h\|_0^2 = b(g_h,\varphi) = b(g_h,\varphi-\pi_2\varphi) + b(g_h,\pi_2\varphi) - a(g_h,\pi_h^*\pi_2\varphi) + a(g_h,\pi_h^*\pi_2\varphi). \tag{5.21}$$

From (4.11) and interpolation theory, we have

$$|b(g_h,\varphi-\pi_2\varphi)| \le C|g_h|_1|\varphi-\pi_2\varphi|_1 \le Ch|g_h|_1\|\varphi\|_2 \le Ch^4|u|_4\|g_h\|_0. \tag{5.22}$$

By Lemma 7

$$|b(g_h,\pi_2\varphi) - a(g_h,\pi_h^*\pi_2\varphi)| \le Ch|g_h|_1|\varphi|_{2,h} \le Ch^4|u|_4|\varphi|_{2,h}. \tag{5.23}$$

By (4.1) and (4.2), we know $a(g_h,\pi_h^*\pi_2\varphi) = -a(u-\pi_2u,\pi_h^*\pi_2\varphi)$. By Lemma 6,

$$|a(g_h,\pi_h^*\pi_2\varphi)| \le Ch^4|u|_{4,h}|\varphi|_{2,h} + Ch^4|u|_{5,h}|\varphi|_{1,h}. \tag{5.24}$$

Substituting (5.22)–(5.24) into (5.21), noting that Lemma 5 and $|\varphi|_{1,h} \le C\|\varphi\|_2$, (5.19) holds and the proof is completed. □

By Lemma 1, $\|u_h - \pi_2u\|_{0,h} \le \frac{5}{2}\|u_h - \pi_2u\|_0$, hence, Theorem 5 tells us that the scheme in this paper has nodal point superconvergence.

6. Numerical example

In this section, a numerical example is provided to verify the effectiveness of the superconvergent alternating direction finite volume element method in Section 3.

Table 1
The discrete L^2 norm errors in the example.

h	QIFVEM-OSP	UQIFVEM	TFVEM	RFVEM
0.025	5.9188×10^{-7}	1.2703×10^{-4}	2.2453×10^{-5}	4.1940×10^{-4}
0.0125	3.7287×10^{-8}	3.1768×10^{-5}	5.6132×10^{-6}	1.0482×10^{-4}
0.00625	2.3402×10^{-9}	7.9381×10^{-6}	1.4004×10^{-6}	2.6201×10^{-5}

Table 2
The discrete L^2 norm errors of gradients at some points in the example.

	$h = 0.0125$		$h = 0.00625$	
	QIFVEM-OSP	UQIFVEM	QIFVEM-OSP	UQIFVEM
Eugosp	2.4830×10^{-5}	1.8796×10^{-4}	3.1038×10^{-6}	4.6746×10^{-5}
Eugc	1.5594×10^{-3}	1.4792×10^{-3}	3.8990×10^{-4}	3.6982×10^{-4}

Table 3
Numerical convergence orders of some schemes.

	QIFVEM-OSP	UQIFVEM	TFVEM	RFVEM
Order of nodal values	3.99	2.00	2.00	2.00
Order of gradients at OSPs	3.00	2.02		
Order of gradients at CEs	2.00	2.00		

Example. In (3.1), let $f(x, y) = (2\pi^2-1)\exp(y)\sin\pi(x+y)-2\pi\exp(y)\cos\pi(x+y)$ and the accurate solution is determined by $u = \exp(y)\sin\pi(x + y)$. It is obvious that $u|_{\partial\Omega} \neq 0$. The example is computed by quadratic interpolation finite volume element method based on optimal stress points (QIFVEM-OSP) (3.11)–(3.12), usual quadratic interpolation finite volume element method (UQIFVEM), usual finite volume element method based on triangular partition (TFVEM) [5] and rectangular partition (RFVEM) [2]. Compared with QIFVEM-OSP, UQIFVEM uses different control volumes. For instance, for nodal point (x_{2i-1}, y_{2j-1}), its control volume is $V_{2i-1,2j-1} = [x_{2i-1} - \frac{1}{2}h, x_{2i-1} + \frac{1}{2}h] \times [y_{2j-1} - \frac{1}{2}h, y_{2j-1} + \frac{1}{2}h]$. For nodal point (x_{2i}, y_{2j}), its control volume is $V_{2i,2j} = [x_{2i-1} + \frac{1}{2}h, x_{2i+1} - \frac{1}{2}h] \times [y_{2j-1} + \frac{1}{2}h, y_{2j+1} - \frac{1}{2}h]$. For simplicity, assume that the partition is uniform with step length h. The discrete L^2 norm errors $\|u - u_h\|_{0,h}$ for different h are shown in Table 1.

We know from Table 1 that QIFVEM-OSP in this paper has truly high accuracy, which is obviously higher than that of the usual linear and quadratic FVEM. Further, we use the numerical results of QIFVEM-OSP and UQIFVEM to compute the gradients at optimal stress points and the center in every element. The discrete L^2 norm errors are shown in Table 2, where $Eugosp = \|\nabla(u - u_h)(\mathrm{OSP})\|$ is defined by the expression in the left hand side of (5.2) and $Eugc = \left(\sum_{i,j=1}^{N} |\nabla(u - u_h)(x_{2i-1}, y_{2j-1})|^2 h^2\right)^{\frac{1}{2}}$.

Table 2 tells us that the gradients of numerical solutions at optimal stress points are more accurate than those in other points and the numerical gradients of QIFVEM-OSP are also more accurate than those of UQIFVEM.

If $\|u - u_h\|_{0,h} \approx Ch^p$, then $-\ln\|u - u_h\|_{0,h} \approx -\ln C + p(-\ln h)$. Thus the slope of the line $-\ln\|u - u_h\|_{0,h}$ against $-\ln h$ represents the numerical convergence order p. For QIFVEM-OSP, using the data in Tables 1 and 2, we have the linear fitting functions

$$-\ln\|u - u_h\|_{0,h} = -0.383957 + 3.99126(-\ln h),$$
$$-\ln Eugosp = -2.54205 + 2.99988(-\ln h),$$
$$-\ln Eugc = -2.29903 + 1.9996(-\ln h).$$

For other schemes, we can also get numerical convergence orders by linear fitting method. We summarize the numerical convergence orders of linear and quadratic finite volume element schemes in Table 3.

From Table 3, we know QIFVEM-OSP has nearly fourth order accuracy at nodal points with respect to discrete L^2 norm, whereas UQIFVEM has second order accuracy, which is same as that of TFVEM and RFVEM. QIFVEM-OSP also gets third order accuracy for numerical gradients at optimal stress points, which is more accurate than that of UQIFVEM.

7. Conclusions

In this paper, we construct a kind of biquadratic finite volume element method for Poisson's equations by restricting the four optimal stress points of biquadratic interpolation as the vertices of a control volume. The method has the following good properties.

- The method has optimal H^1 norm error estimate $O(h^2)$.
- The method has L^2 norm error estimate $O(h^3)$.

- The numerical gradients of the method have $O(h^3)$ superconvergence order at optimal stress points.
- The method has fourth order computational accuracy at nodal points with respect to discrete L^2 norm, that is, the method has displacement superconvergence.

In practical computation, we can employ cubic or quartic interpolations to recover the function values or derivatives at arbitrary points in the given domain.

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